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Mass in anti-de Sitter spaces

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Abstract

The boundary stress tensor approach has proven extremely useful in defining mass and angular momentum in asymptotically anti-de Sitter spaces with CFT duals. An integral part of this method is the use of boundary counterterms to regulate the gravitational action and stress tensor. In addition to the standard gravitational counterterms, in the presence of matter we advocate the use of a finite counterterm proportional to ϕ^2 (in five dimensions). We demonstrate that this finite shift is necessary to properly reproduce the expected mass/charge relation for R-charged black holes in AdS_5.

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1 Introduction

While the notion of mass is perhaps intuitively obvious, much of this intuition is related to flat space, where mass may be used to label representations of the Poincaré group. Once we consider curved space, some of this intuition falls apart. Of course, the idea of mass as a source of curvature is an essential component of general relativity. Nevertheless, in the absence of Poincaré symmetry, mass can no longer be defined in a straightforward manner.

In fact, in a closed spacetime, there can be no intrinsic meaning to the mass of the universe in much the same way as there cannot be any net charge in a closed space. On the other hand, there has been a long history of defining mass for spaces with an asymptotic region. Perhaps one of the best known prescriptions is that of Arnowitt, Deser and Misner (ADM), which may be most straightforwardly applied in asymptotically flat spacetimes. This is essentially equivalent to reading off the mass from the Newtonian potential, $\Phi(r) \sim -M/r$, where $\Phi(r)$ may may be extracted from the time-time component of the metric, $g_{tt} \sim -(1 + 2\Phi(r))$.

In general, the ADM prescription can also be applied to spacetimes with non-flat asymptotic regions, such as asymptotically anti-de Sitter (AdS) spaces [1]. In such cases, the mass may be extracted by comparison to a reference (*e.g.* vacuum AdS) background. However, care must be taken to ensure that the deviation from the reference background is sufficiently well controlled. This task is often made difficult in practice because one must control the reparametrization invariance of both deformed and undeformed backgrounds to ensure a well defined result. A similar approach to mass has been taken by Brown and York [2] in defining a quasilocal stress tensor through the variation of the gravitational action

$$T^{ab} = \frac{2}{\sqrt{-h}} \frac{\delta S}{\delta h_{ab}}, \tag{1.1}$$

where h_{ab} is the boundary metric. In general, T^{ab} diverges as the boundary is pushed off to infinity, and hence a background subtraction is again necessary.

More recently, an alternative procedure has been demonstrated where the boundary stress tensor may be regulated by the introduction of appropriate boundary counterterms [3]. The advantage of this method is that the regulated gravitational action and resulting boundary stress tensor may be obtained directly for the background at hand, without having to introduce a somewhat artificial reference background. This counterterm method has become quite standard when applied to AdS/CFT, as the boundary counterterms have a natural interpretation as conventional field theory counterterms that show up in the dual CFT.

In general, it is only necessary to introduce a handful of boundary counterterms in order to cancel divergences in the gravitational action. For example, in AdS_5, only two counterterms are necessary. However, one could equally well add in an arbitrary amount of *finite* counterterms. While this would certainly change the values of the action integral and corresponding boundary stress tensor, this has a natural interpretation in the dual CFT as simply the usual freedom to change renormalization prescriptions.

Although one is in principle free to choose any desired prescription, some are perhaps better motivated than others. For example, in a gauge theory, one tends to avoid non-gauge invariant regulators, and in supersymmetric theories, one generally chooses a 'supersymmetric' scheme. While the introduction of finite counterterms has often been overlooked, this can lead to somewhat surprising results. In particular, it was shown in [4] that, in the absence of finite counterterms, single R-charged black holes in AdS_5 obey a non-linear mass/charge relation, $M \sim \frac{3}{2}\mu + q - \frac{1}{3}g^2q^2$, where μ is the non-extremality parameter and $g = 1/\ell$ is the inverse AdS radius. While nothing prevents us from taking this as a definition of mass, it nevertheless appears to be in conflict with the BPS expectation that $M \geq |q|$.

In this paper, we propose to include a finite counterterm related to the scalar fields, and in doing so will recover the expected linear relation $M \sim \frac{3}{2}\mu + q$. We also demonstrate that for three-charge AdS_5 black holes in the STU model, the mass/charge relation remains linear, namely $M \sim \frac{3}{2}\mu + q_1 + q_2 + q_3$. The boundary stress tensor method can also be applied to the newly constructed Gutowski-Reall black holes in AdS_5 [5, 6]. We compute the masses of these solutions and demonstrate equivalence with the results obtained in [6] using the Ashtekar and Das approach [7].

We begin in section 2 with a review of the boundary counterterm procedure. While this is by now familiar, we find it useful here to set the notation and prepare the groundwork for the subsequent calculations. In section 3 we include matter fields (general scalars and vectors) and in section 4 we complete the regulation procedure by introducing a finite ϕ^2 counterterm. We verify in section 5 that this counterterm results in the linear mass relation mentioned above. Finally, we examine the Gutowski-Reall black holes in section 6 and conclude in section 7.

2 The stress tensor for pure gravity

Before considering the matter coupled system, we briefly review the boundary counterterm method for a purely gravitational theory [3]. We work in five dimensions with a negative cosmological constant, so that the Einstein action may be written as

$$S[g_{\mu\nu}] = S_{\mathrm{bulk}} + S_{\mathrm{GH}} = -\frac{1}{16\pi G_5}\int_{\mathcal{M}} d^5x\sqrt{-g}\left[R + 12g^2\right] + \frac{1}{8\pi G_5}\int_{\partial\mathcal{M}} d^4x\sqrt{-h}\;\Theta, \tag{2.1}$$

where g is the inverse radius of AdS_5. The Gibbons-Hawking surface term is included to ensure a proper variational principle for a spacetime $\mathcal{M}$ with boundary $\partial\mathcal{M}$. Here, Θ is the trace of the extrinsic curvature $\Theta^{\mu\nu}$ of the boundary, defined by

$$\Theta^{\mu\nu} = -\frac{1}{2}\left(\nabla^\mu n^\nu + \nabla^\nu n^\mu\right), \tag{2.2}$$

where n^μ is the outward-pointing normal on $\partial\mathcal{M}$.

In the holographic context, it is natural to single out a radial coordinate r, and thus we decompose the bulk five-dimensional metric according to

$$ds_5^2 = N^2dr^2 + h_{ab}\big(dx^a + V^adr\big)\big(dx^b + V^bdr\big). \tag{2.3}$$

This is essentially an ADM decomposition, except that here the radial coordinate r plays the rôle of time. Furthermore we will choose r so that the boundary, $\partial\mathcal{M}$, is reached as $r\to\infty$. The four-dimensional metric h_{ab} then represents the induced metric on $\partial\mathcal{M}$. Following [2], the quasi-local stress tensor on the surface $\partial\mathcal{M}$ is then defined through the variation of the gravitational action with respect to the boundary metric h_{ab}

$$T^{ab}=\frac{2}{\sqrt{-h}}\frac{\delta S}{\delta h_{ab}}=\frac{1}{8\pi G_5}\left(\Theta^{ab}-\Theta h^{ab}\right). \tag{2.4}$$

Given T^{ab}, it is possible to extract the ADM mass and momentum as appropriate conserved quantities. To do so, we foliate the boundary spacetime $\partial\mathcal{M}$ by spacelike surfaces Σ with metric $\sigma_{\alpha\beta}$, so that

$$ds_4^2\equiv h_{ab}dx^a dx^b=-N_\Sigma^2dt^2+\sigma_{\alpha\beta}\big(dx^\alpha+V_\Sigma^\alpha dt\big)\big(dx^\beta+V_\Sigma^\beta dt\big). \tag{2.5}$$

The conserved charges are then obtained by integrating the time component of the conserved stress tensor over the three-dimensional surface Σ. More precisely, for an isometry of the boundary geometry generated by a Killing vector ξ^a, the corresponding conserved charge is given by

$$Q_\xi=\int_\Sigma dx^3\sqrt{\sigma}\left(u^aT_{ab}\xi^b\right), \tag{2.6}$$

where u^a is the timelike unit normal to the surface Σ. For a time-translationally invariant spacetime, we take the Killing vector to be $\xi^a=N_\Sigma u^a$, in which case the conserved charge Q_ξ corresponds to the total energy of the spacetime.

In general, it can be shown that the stress tensor defined in this matter (as well as the on-shell value of the action) diverges when the surface $\partial\mathcal{M}$ is pushed to infinity. While [2] removes this divergence through background subtraction, the method of [3] is to instead regulate the action, (2.1), through the addition of boundary counterterms, $S_{\rm ct}[h_{ab}]$. This also yields a counterterm addition to the stress tensor,

$$T^{ab}_{\rm reg}=\frac{1}{8\pi G_5}(\Theta^{ab}-\Theta h^{ab})+\frac{2}{\sqrt{-h}}\frac{\delta S_{ct}}{\delta h_{ab}}. \tag{2.7}$$

Only two counterterms, of the forms

$$S_1=\frac{1}{8\pi G_5}\int d^4x\sqrt{-h},\qquad S_2=\frac{1}{8\pi G_5}\int d^4x\sqrt{-h}\mathcal{R}, \tag{2.8}$$

are necessary for regulating the divergences of the gravitational action, (2.1). Here $\mathcal{R}$ is the scalar curvature of the boundary metric (2.5). The resulting action has the form

$$S_{\rm reg}=S_{\rm bulk}+S_{\rm GH}+3gS_1+(4g)^{-1}S_2. \tag{2.9}$$

In addition, the counterterms contribute

$$T_1^{ab}=\frac{1}{8\pi G_5}h^{ab},\qquad T_2^{ab}=\frac{1}{8\pi G_5}(2\mathcal{R}^{ab}-\mathcal{R}h^{ab}), \tag{2.10}$$

to the regulated stress tensor.

2.1 Mass of the Schwarzschild-AdS spacetimes

To illustrate the above general discussion, we review the case of Schwarzschild-AdS$_5$, which has attracted much previous attention as the spacetime corresponding to non-extremal D3-branes. In five dimensions, the metric may be written as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2, \tag{2.11}$$

where $f(r) = 1 - (r_0/r)^2 + g^2r^2$. We implicitly define r_+ to be the location of the horizon, given by $f(r_+) = 0$.

When evaluated on shell, the bulk action may be re-expressed in terms of a surface integral. We find

$$I_{\text{bulk}} = \frac{\beta\omega_3}{8\pi G_5}\left(r^2(f-1) + r_+^2\right), \tag{2.12}$$

where we use I to denote the value of the Euclidean action integral. Here, $\beta = 2\pi/T$ is the periodicity along the time direction and $\omega_3 = 2\pi^2$ is the volume of the unit 3-sphere. Note that, in the absence of matter, the action integral (2.12) is easily obtained through the substitution of the trace of the Einstein equation, $R = -20g^2$, into S_{bulk} to obtain

$$I_{\text{bulk}} = -\frac{1}{16\pi G_5}\int d^5x\sqrt{-g}[-8g^2] = \frac{\beta\omega_3}{8\pi G_5}g^2(r^4 - r_+^{(4)}), \tag{2.13}$$

which is equivalent to (2.12) when the expression for $f(r)$ is taken into account. However, as shown in the following section, the expression (2.12) is more general, and continues to hold when matter is added to the system.

In addition to I_{bulk}, the Gibbons-Hawking boundary term gives the contribution

$$I_{\text{GH}} = -\frac{\beta\omega_3}{8\pi G_5}\left(\tfrac{1}{2}r^3f' + 3r^2f\right). \tag{2.14}$$

Thus the complete action is given by

$$I_{\text{GH}} + I_{\text{bulk}} = \frac{\beta\omega_3}{8\pi G_5}\left(-3r^2 + r_+^2 - 3g^2r^4 + r_0^2\right), \tag{2.15}$$

where we have substituted in the explicit form of f.

While the on-shell action diverges like r^4 as we approach the boundary $r \to \infty$, this divergence is removed by the addition of the counterterms (2.8). The appropriately regulated action, (2.9), is given by [3, 8]

$$I_{\text{reg}} = \frac{\beta\pi}{4G_5}\left(r_+^2 - \frac{1}{2}r_0^2 + \frac{3}{8g^2}\right), \tag{2.16}$$

and remains finite. Likewise, the counterterms also lead to a finite stress tensor. Using (2.7) and (2.10), one finds

$$T_{tt} = \frac{1}{8\pi G_5 r^2}\left(\frac{3}{8g} + \frac{3gr_0^2}{2}\right), \tag{2.17}$$

resulting the familiar Schwarzschild-AdS$_5$ energy [3]

$$E = \frac{3\pi}{8G_5}\left(r_0^2 + \frac{1}{4g^2}\right), \tag{2.18}$$

which naturally includes the CFT Casimir energy in addition to the non-extremality parameter r_0.

3 Addition of the matter sector

In order to examine the mass of charged black hole solutions, we must first extend the standard counterterm procedure by introducing a matter sector to the bulk action. Since the action is no longer that of pure gravity, it may now be necessary to include additional local counterterms on $\partial\mathcal{M}$ constructed out of the boundary values of the matter fields in order to cancel all divergences.

Although we eventually turn to solutions of gauged $\mathcal{N}=2$ supergravity in five dimensions, we first consider the a general matter coupled gravity system with action

$$\begin{aligned} S[g_{\mu\nu},\phi^i,A_\mu^I] &= -\frac{1}{16\pi G_5}\int_{\mathcal{M}} d^5x\sqrt{-g}[R - \tfrac{1}{2}g(\phi)_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - \tfrac{1}{4}G_{IJ}(\phi)F_{\mu\nu}^I F^{\mu\nu\,J} - V(\phi)] \\ &\quad + \frac{1}{8\pi G_5}\int_{\partial\mathcal{M}} d^4x\sqrt{-h}\;\Theta. \end{aligned} \tag{3.1}$$

To evaluate the on-shell value of the bulk action, we note that the Einstein equation, written in Ricci form, is given by

$$R_{\mu\nu} = \tfrac{1}{2}g(\phi)_{ij}\partial_\mu\phi^i\partial_\nu\phi^j + \tfrac{1}{2}G_{IJ}(F_{\mu\lambda}^I F_\nu{}^{\lambda\,J} - \tfrac{1}{6}g_{\mu\nu}F_{\rho\sigma}^I F^{\rho\sigma\,J}) + \tfrac{1}{3}g_{\mu\nu}V. \tag{3.2}$$

Taking the trace of this equation to obtain R, and substituting it into the action integral gives

$$I_{\rm bulk} = -\frac{1}{16\pi G_5}\int d^5x\sqrt{-g}[-\tfrac{1}{6}G_{IJ}F_{\mu\nu}^I F^{\mu\nu\,J} + \tfrac{2}{3}V]. \tag{3.3}$$

While this is a simplification of the action integral, it appears to be as far as we may proceed without further input. Thus we now focus on static electrically charged black hole solutions, and take an ansatz of the form

$$\begin{aligned} ds^2 &= -e^{-4B(r)}f(r)dt^2 + e^{2B(r)}\left(\frac{dr^2}{f(r)} + r^2 d\Omega_3^2\right), \\ \phi^i &= \phi^i(r), \qquad A_t^I = A_t^I(r), \end{aligned} \tag{3.4}$$

where the 3-sphere may be parametrized as

$$d\Omega_3^2 = d\psi^2 + \sin^2\psi\, d\Omega_2^2. \tag{3.5}$$

In this case, the $R_{\psi\psi}$ component of the Einstein equation, (3.2), yields

$$2R^{\psi}_{\psi} = -\tfrac{1}{6}G_{IJ}F^{I}_{\mu\nu}F^{\mu\nu\,J} + \tfrac{2}{3}V, \tag{3.6}$$

which has the same form as the integrand of (3.3). This gives a simple result for the action integral

$$I_{\text{bulk}} = -\frac{1}{8\pi G_5}\int d^5x\sqrt{-g}R^{\psi}_{\psi}, \tag{3.7}$$

provided we follow the ansatz (3.4). Working out the $R_{\psi\psi}$ component explicitly, we obtain

$$\begin{aligned} I_{\text{bulk}} &= \frac{\beta\omega_3}{8\pi G_5}\int dr\frac{d}{dr}\left[r^3 fB' + r^2(f-1)\right] \\ &= \frac{\beta\omega_3}{8\pi G_5}\left(r^3 fB' + r^2(f-1) + r_+^2\right), \end{aligned} \tag{3.8}$$

where in the last line we have taken the range of r to be from the horizon r_+ to the finite but large value r where we cut off the space.

To evaluate the Gibbons-Hawking surface term, we start with the unit normal in the r direction, $n^r = e^{-B}f^{\frac{1}{2}}$. Evaluating its divergence yields

$$\Theta = -\nabla_\mu n^\mu = -e^{-B}f^{\frac{1}{2}}\left(B' + \frac{f'}{2f} + \frac{3}{r}\right), \tag{3.9}$$

so that

$$I_{\text{GH}} = -\frac{\beta\omega_3}{8\pi G_5}\left(r^3 fB' + \tfrac{1}{2}r^3 f' + 3r^2 f\right). \tag{3.10}$$

Curiously, the B' dependent terms in I_{bulk} and I_{GH} cancel when added together. We find

$$I_{GH} + I_{bulk} = \frac{\beta\omega_3}{8\pi G_5}\left(-2r^2 f - \tfrac{1}{2}r^3 f' - r^2 + r_+^2\right). \tag{3.11}$$

This expression as it stands is divergent, and must be regulated by an appropriate counterterm subtraction. However, we emphasize that this expression includes all effects of the scalars and gauge fields of (3.1), although they do not show up explicitly here. It is remarkable that the unregulated action only depends explicitly on the 'blackening function' $f(r)$ in (3.4). However, as a solution to the Einstein equation, f naturally includes residual information of all appropriate scalar and gauge charges.

For the metric ansatz, (3.4), the boundary counterterms S_1 and S_2 take the simple form

$$I_1 = \frac{\beta\omega_3}{8\pi G_5}r^3 f^{\frac{1}{2}}e^{B}, \qquad I_2 = \frac{3\beta\omega_3}{4\pi G_5}rf^{\frac{1}{2}}e^{-B}, \tag{3.12}$$

so that the regulated action integral, (2.9), is given by

$$I_{\text{reg}} = \frac{\beta\omega_3}{8\pi G_5}\left(-2r^2 f - \tfrac{1}{2}r^3 f' - r^2 + r_+^2 + 3gr^3 f^{\frac{1}{2}}e^{B} + \tfrac{3}{2}g^{-1}rf^{\frac{1}{2}}e^{-B}\right). \tag{3.13}$$

Although this is not manifestly finite, we demonstrate explicitly that it is indeed so for R-charged black holes in AdS$_5$.

3.1 R-charged black holes in AdS$_5$

We now turn to the examination of R-charged black holes. These electrically charged black holes are static stationary solutions to gauged $\mathcal{N} = 2$ supergravity, and have a metric of the form [9]

$$ds^2 = -\mathcal{H}^{-2/3} f dt^2 + \mathcal{H}^{1/3} \left(\frac{dr^2}{f} + r^2 d\Omega_3^2 \right), \tag{3.14}$$

where

$$f = 1 - \frac{r_0^2}{r^2} + g^2 r^2 \mathcal{H}. \tag{3.15}$$

The 'harmonic function' $\mathcal{H}$ is related to e^{2B} of the ansatz (3.4) by $\mathcal{H} = e^{6B}$. In the STU model, $\mathcal{H}$ is given by the product of three harmonic functions

$$\mathcal{H} = H_1 H_2 H_3 = \left(1 + \frac{q_1}{r^2}\right)\left(1 + \frac{q_2}{r^2}\right)\left(1 + \frac{q_3}{r^2}\right). \tag{3.16}$$

However, in general, we still expect $\mathcal{H}$ to have a large r expansion of the form

$$\mathcal{H} = 1 + \frac{Q^{(1)}}{r^2} + \frac{Q^{(2)}}{r^4} + \frac{Q^{(3)}}{r^6} + \cdots. \tag{3.17}$$

It is now straightforward to substitute the metric functions $f(r)$ and $\mathcal{H}(r)$ into the regulated action integral, (3.13). Up to terms that vanish in the limit $r \to \infty$, we obtain the finite expression

$$I_{\text{reg}} = \frac{\beta\pi}{4G_5}\left(r_+^2 - \frac{1}{2} r_0^2 + \frac{3}{8g^2} - g^2\left(\frac{Q^{(1)\,2}}{3} - Q^{(2)} \right) \right). \tag{3.18}$$

This expression is the generalization of (2.16) to the case of R-charged black holes, where the charges are given by $Q^{(1)} = \sum_i q_i$ and $Q^{(2)} = \sum_{i<j} q_i q_j$. Note that this expression is obtained directly from the metric (3.14), without even specifying the gauge fields and scalars associated with the solution.

As can be seen from (3.18), the black hole charges enter non-linearly in the action integral. In particular, for the single charged black hole ($Q^{(2)} = 0$) this expression reduces to that derived previously in [4]. Although there is nothing inherently wrong with the nonlinear charge behavior, it is somewhat unexpected, especially considering that it remains nonlinear in the BPS limit. Of course, these black holes actually become singular in the limit. But nevertheless, the formal BPS expression could have been expected to hold. Indeed, it turns out that there is a simple means of removing the nonlinearity in (3.18) through the introduction of finite boundary counterterms. This is what we now proceed to demonstrate.

4 Addition of finite counterterms

In general, boundary counterterms have been introduced as a means of regulating divergences in the gravitational action. However, we wish to emphasize here that nothing prevents us from introducing *finite* counterterms as well. Such expressions yield a finite renormalization of the gravitational action, and are hence dual to finite shifts in the renormalization of the CFT. As a result, they are may be viewed as generating shifts between different renormalization prescriptions of the CFT.

Since we have introduced the matter fields ϕ^i and A^I_μ into the action (3.1), it is natural to construct local boundary counterterms such as

$$\begin{aligned}
S_{\phi^2} &= \frac{1}{8\pi G_5}\int_{\partial\mathcal{M}} d^4x\sqrt{-h}\, g_{ij}\phi^i\phi^j, \qquad S_{\partial\phi^2} = \frac{1}{8\pi G_5}\int_{\partial\mathcal{M}} d^4x\sqrt{-h}\, g_{ij}\partial_a\phi^i\partial^a\phi^j, \\
S_{F^2} &= \frac{1}{8\pi G_5}\int_{\partial\mathcal{M}} d^4x\sqrt{-h}\, G_{IJ}F^I_{ab}F^{ab\,J},
\end{aligned} \tag{4.1}$$

where the a, b indices correspond to the boundary surface $\partial\mathcal{M}$. In particular, radial ∂_r derivatives are absent, as they are not local to the boundary.

For the spherically symmetric black holes, the fields are only functions of the radial coordinate r. Hence the two-derivative counterterms in (4.1) will not contribute. As a result, we consider only I_{ϕ^2}, which takes the form

$$I_{\phi^2} = \frac{\beta\omega_3}{8\pi G_5} r^3 f^{\frac{1}{2}} e^B (g_{ij}\phi^i\phi^j). \tag{4.2}$$

So far, the analysis has been completely general, at least for this class of spherically symmetric and stationary solutions. However, at this stage, it is necessary to provide the explicit asymptotic form of the scalars corresponding to the black hole metric of (3.14).

To proceed, we consider the specific example of the STU model. Here, there are three $U(1)$ gauge fields and two scalars, with the scalars defined by

$$\begin{aligned}
X^1 &= e^{-\frac{1}{\sqrt{6}}\phi_1 - \frac{1}{\sqrt{2}}\phi_2} = H_1^{-1}\mathcal{H}^{\frac{1}{3}}, \\
X^2 &= e^{-\frac{1}{\sqrt{6}}\phi_1 + \frac{1}{\sqrt{2}}\phi_2} = H_2^{-1}\mathcal{H}^{\frac{1}{3}}, \\
X^3 &= e^{\frac{2}{\sqrt{6}}\phi_1} = H_3^{-1}\mathcal{H}^{\frac{1}{3}},
\end{aligned} \tag{4.3}$$

where $\mathcal{H} = H_1H_2H_3$, so that $X^1X^2X^3 = 1$. The two independent scalars ϕ_1 and ϕ_2 may be re-expressed as

$$\phi_1 = \frac{1}{\sqrt{6}}\left(\log H_1 + \log H_2 - 2\log H_3\right), \tag{4.4}$$

$$\phi_2 = \frac{1}{\sqrt{2}}\left(\log H_1 - \log H_2\right). \tag{4.5}$$

This gives in turn

$$\vec{\phi}^{\,2} = \phi_1^2 + \phi_2^2 = \frac{1}{r^4}\left(\frac{2}{3}Q^{(1)\,2} - 2Q^{(2)}\right) + \cdots. \tag{4.6}$$

Substituting this into (4.2) yields a finite contribution

$$\frac{1}{g}I_{\phi^2} = \frac{\beta\pi}{4G_5}\left(\frac{2}{3}Q^{(1)\,2} - 2Q^{(2)}\right), \tag{4.7}$$

which has the exact same charge dependence as the finite part of the action, (3.18). As a result, it may be used as a finite counterterm to completely cancel the charge dependence of the action. The regulated action integral, including finite counterterm

$$I_{\rm reg} + \frac{g}{2}I_{\phi^2} = \frac{\beta\pi}{4G_5}\left(r_+^2 - \frac{1}{2}r_0^2 + \frac{3}{8g^2}\right), \tag{4.8}$$

is then identical to that of the Schwarzschild-AdS solution, (2.16). In fact, we are advocating the use of the full counterterm action

$$I_{\rm complete} = I_{\rm bulk} + I_{\rm GH} + 3gI_1 + \frac{1}{4g}I_2 + \frac{g}{2}I_{\phi^2}, \tag{4.9}$$

for black holes in AdS_5 with or without R charge. For the latter, of course, the I_{ϕ^2} counterterm vanishes. However, we may view this counterterm action as universal, with all coefficients independent of charge.

5 The regulated boundary stress tensor

In the previous section, we have shown that an appropriate counterterm prescription exists for five-dimensional R-charged black holes that preserves the standard expression (2.16) for the action integral independent of charge. We now turn to the calculation of the boundary stress tensor and the extraction of the ADM energy.

We begin with the unregulated stress tensor, (2.4), given by

$$T^{ab} = \frac{1}{8\pi G_5}\left(\Theta^{ab} - \Theta h^{ab}\right). \tag{5.1}$$

For the metric (3.4), the extrinsic curvature takes the form

$$\begin{aligned}\Theta^{tt} &= -\left(-2B' + \frac{f'}{2f}\right)h^{tt}e^{-B}f^{\frac{1}{2}},\\ \Theta^{\alpha\beta} &= -\left(B' + \frac{1}{r}\right)h^{\alpha\beta}e^{-B}f^{\frac{1}{2}},\end{aligned} \tag{5.2}$$

so that

$$\Theta = -\left(B' + \frac{3}{r} + \frac{f'}{2f}\right)e^{-B}f^{\frac{1}{2}}. \tag{5.3}$$

Substituting these expressions into (5.1) gives

$$T_{tt} = \frac{g}{8\pi G_5}\left(-3g^2r^2 - Q^{(1)}g^2 - \frac{9}{2} + \frac{1}{r^2}\left(\frac{9r_0^2}{2} + 3Q^{(1)} - \frac{9}{8g^2}\right)\right). \tag{5.4}$$

At the same time, the local gravitational counterterms, S_1 and S_2, give rise to the contribution

$$\begin{aligned}\tilde{T}_{tt} &= -\frac{g}{8\pi G_5} 3g_{tt}\left(1+\frac{e^{-2B}}{2g^2r^2}\right) \qquad (5.5)\\ &= \frac{g}{8\pi G_5}\left(3g^2r^2+Q^{(1)}g^2+\frac{9}{2}+\frac{1}{r^2}\left(-2Q^{(1)}-3r_0^2+g^2\left(Q^{(2)}-\frac{1}{3}Q^{(1)\,2}\right)+\frac{3}{2g^2}\right)\right),\end{aligned}$$

so that the gravitationally regulated value of T_{tt} is

$$T_{tt}^{\mathrm{reg}} = \frac{g}{8\pi G_5 r^2}\left(Q^{(1)}+\frac{3r_0^2}{2}+g^2\left(Q^{(2)}-\frac{Q^{(1)\,2}}{3}\right)+\frac{3}{8g^2}\right). \qquad (5.6)$$

While this expression yields a finite energy when inserted in (2.6), the term quadratic in charge gives rise to a non-linear mass/charge relation, as first noted in [4]. In fact, setting $Q^{(2)}=0$ reproduces the single-charge black hole result of [4].

Of course, the introduction of the finite counterterm S_{ϕ^2} also shifts the stress tensor according to

$$T_{\phi^2}^{ab} = \frac{1}{8\pi G_5} h^{ab}(g_{ij}\phi^i\phi^j). \qquad (5.7)$$

For the STU model, the evaluation of $(g_{ij}\phi^i\phi^j)$ follows from (4.6). Including $T_{\phi^2}^{ab}$ results in cancellation of the nonlinear charge term, so that the fully regulated value of T_{tt} takes on the simple form

$$T_{tt}^{\mathrm{complete}} = \frac{g}{8\pi G_5 r^2}\left(Q^{(1)}+\frac{3r_0^2}{2}+\frac{3}{8g^2}\right). \qquad (5.8)$$

Therefore the energy is given by

$$\begin{aligned}E &= \frac{\pi}{4G_5}\left(Q^{(1)}+\frac{3r_0^2}{2}+\frac{3}{8g^2}\right)\\ &= \frac{3\pi}{8G_5}\left(r_0^2+\frac{2}{3}q_1+\frac{2}{3}q_2+\frac{2}{3}q_3+\frac{1}{4g^2}\right), \qquad (5.9)\end{aligned}$$

where in the last line we have explicitly written out the three charges of the STU model. This energy generalizes the expression (2.18) to the case of charged black holes in AdS$_5$.

By adding a finite counterterm, I_{ϕ^2}, we have been able to provide a rigorous justification of the AdS$_5$ black hole mass originally given in [9]. We believe this expression is natural from the viewpoint of thermodynamics, in that the three independent charges of the STU model (or equivalently the three commuting $U(1)^3 \subset SO(6)_R$ charges of the four-dimensional $\mathcal{N}=4$ theory) contribute linearly to the mass in (5.9). Note that the non-linear term $Q^{(2)} - Q^{(1)\,2}/3$ vanishes identically for the three equal charge black hole. In this case, either mass expression yields the same result. In fact, this must be true; since this black hole may be viewed as a solution of the pure five-dimensional $\mathcal{N}=2$ supergravity, and there are no scalars in this theory, the scalar counterterm cannot contribute to the mass.

6 Gutowski-Reall Solutions

Another example where the importance of the S_{ϕ^2} counterterm shows up is in the case of the recently constructed Gutowski-Reall supersymmetric black hole solutions [5, 6]. Unlike the stationary R-charged black holes investigated above which become singular in the BPS limit, the Gutowski-Reall solutions maintain a regular horizon through non-zero angular momentum.

The general rotating solution has a metric of the form [6]

$$ds^2 = -f^2dt^2 - 2f^2w\,dt\,\sigma_L^3 + f^{-1}g^{-1}dr^2 + \frac{r^2}{4}\left[f^{-1}\left(\left(\sigma_L^1\right)^2 + \left(\sigma_L^2\right)^2\right) + f^2h\left(\sigma_L^3\right)^2\right], \quad (6.1)$$

where the functions f, g, w and h are

$$\begin{aligned} f &= \left(1 + \frac{\alpha_1}{r^2} + \frac{\alpha_2}{r^4} + \frac{\alpha_3}{r^6}\right)^{-\frac{1}{3}}, \qquad g = \left(1 + \frac{\alpha_1}{\ell^2} + \frac{r^2}{\ell^2}\right), \\ w &= -\frac{\epsilon r^2}{2\ell}\left(1 + \frac{\alpha_1}{r^2} + \frac{\alpha_2}{2r^4}\right), \qquad h = f^{-3}g - \frac{4}{r^2}w^2. \end{aligned} \quad (6.2)$$

Here, σ_L^i are right-invariant 1-forms on SU(2) given by

$$\begin{aligned} \sigma_L^1 &= \sin\phi\,d\theta - \cos\phi\sin\theta\,d\psi, \\ \sigma_L^2 &= \cos\phi\,d\theta + \sin\phi\sin\theta\,d\psi, \\ \sigma_L^3 &= d\phi + \cos\theta\,d\psi. \end{aligned} \quad (6.3)$$

In addition, the scalars and gauge fields are given generally by

$$\begin{aligned} X_I &= f\left(\bar{X}_I + \frac{q_I}{r^2}\right), \\ A^I &= fX^Idt + \left(U^I + fwX^I\right)\sigma_L^3, \\ U^I &= \frac{9\epsilon}{4\ell}C^{IJK}\bar{X}_J\left(\bar{X}_Kr^2 + 2q_K\right). \end{aligned} \quad (6.4)$$

To avoid confusion over the gauge coupling g versus the function $g(r)$, we maintain the notation of [6] where ℓ denotes the AdS$_5$ radius. These expressions simplify for the STU model, in which case

$$f = (H_1H_2H_3)^{-1/3}, \qquad X_I = \tfrac{1}{3}H_If, \qquad H_I = 1 + \frac{\mu_I}{r^2}. \quad (6.5)$$

Note that

$$\alpha_1 = \mu_1 + \mu_2 + \mu_3, \qquad \alpha_2 = \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3, \qquad \alpha_3 = \mu_1\mu_2\mu_3, \quad (6.6)$$

are analogous to the $Q^{(i)}$ of (3.17).

Even in the STU model, where the scalar and gauge field behavior is explicit, the analysis is somewhat complicated by rotation. Foliating the boundary metric according to (2.5), we first rewrite (6.1) as

$$ds^2 = -\frac{g}{fh}dt^2 + \frac{dr^2}{fg} + \frac{r^2}{4f}\left[(\sigma_L^1)^2 + (\sigma_L^2)^2 + f^3 h(\sigma_L^3 - \frac{4}{r^2}\frac{w}{h}dt)^2\right]. \tag{6.7}$$

This then allows us to introduce a natural vielbein basis

$$\begin{aligned}
&e_0 = g^{\frac{1}{2}}(fh)^{-\frac{1}{2}}dt, \qquad e_4 = (fg)^{-\frac{1}{2}}dr, \\
&e_1 = \frac{r}{2}f^{-\frac{1}{2}}\sigma_L^1, \qquad e_2 = \frac{r}{2}f^{-\frac{1}{2}}\sigma_L^2, \qquad e_3 = \frac{r}{2}fh^{\frac{1}{2}}(\sigma_L^3 - \frac{4}{r^2}\frac{w}{h}dt).
\end{aligned} \tag{6.8}$$

Given this solution, we may compute the regulated action integral (4.9) as well as the corresponding boundary stress tensor.

6.1 The action integral

While computation of the bulk action is in principle straightforward, the simplification of (3.7) no longer follows due to the rotation. Of course, one may still evaluate I_{bulk} directly from (3.3) and explicit knowledge of the solution. Alternatively, one can use the R_{11} component of the Einstein equation to rewrite (3.7) as

$$I_{\text{bulk}} = -\frac{1}{8\pi G_5}\int d^5x\sqrt{-g}[R_{11} - \tfrac{1}{2}G_{IJ}F_{12}^I F_{12}^J], \tag{6.9}$$

where we have also used the fact that the only non-vanishing vielbein components of the field strength are F_{04}^I, F_{12}^I and F_{34}^I.

For the STU model, the second term in (6.9) has the form

$$G_{IJ}F_{12}^I F_{12}^J = \frac{f^4}{\ell^2 r^8}(3\alpha_2^2 - 8\alpha_1\alpha_3). \tag{6.10}$$

In addition, the Ricci component R_{11} may be written as

$$R_{11} = \frac{f}{r^3}\left[\frac{d}{dr}\left(\frac{1}{2}r^3 g\frac{f'}{f} + r^2(2-g)\right) - 2rhf^3\right]. \tag{6.11}$$

Combining these expressions and integrating from the horizon $r = 0$ to a large radial value r yields

$$\begin{aligned}
I_{\text{bulk}} \;=\; &\frac{\beta\omega_3}{8\pi G_5}\Big[\frac{r^4}{\ell^2} + \frac{2\alpha_1 r^2}{3\ell^2} + \frac{\alpha_1}{3} + \frac{2\alpha_2}{3\ell^2} + \frac{1}{\ell^2}\Big(\frac{\mu_1^2(\mu_2-\mu_3)^2 + \mu_2^2\mu_3^2}{(\mu_1-\mu_2)(\mu_1-\mu_3)}\log\mu_1 \\
&+ \frac{\mu_2^2(\mu_1-\mu_3)^2 + \mu_1^2\mu_3^2}{(\mu_2-\mu_3)(\mu_2-\mu_3)}\log\mu_2 + \frac{\mu_3^2(\mu_1-\mu_2)^2 + \mu_1^2\mu_2^2}{(\mu_3-\mu_1)(\mu_3-\mu_2)}\log\mu_3\Big)\Big].
\end{aligned} \tag{6.12}$$

Note the appearance of the logarithmic terms that were not present in the non-rotating case.

In anticipation of the computation of the boundary stress tensor, we find that the non-vanishing components of the extrinsic curvature are

$$\theta_{00} = \tfrac{1}{2}(fg)^{\frac{1}{2}}\left(-\frac{f'}{f}+\frac{g'}{g}-\frac{h'}{h}\right), \qquad \theta_{03} = \frac{f^2 w}{r}\left(\frac{2}{r}+\frac{h'}{h}-\frac{w'}{w}\right),$$

$$\theta_{11} = \theta_{22} = \tfrac{1}{2}(fg)^{\frac{1}{2}}\left(-\frac{2}{r}+\frac{f'}{f}\right), \qquad \theta_{33} = \tfrac{1}{2}(fg)^{\frac{1}{2}}\left(-\frac{2}{r}-\frac{2f'}{f}-\frac{h'}{h}\right), \quad (6.13)$$

so that

$$\theta = \tfrac{1}{2}(fg)^{\frac{1}{2}}\left(-\frac{6}{r}+\frac{f'}{f}-\frac{g'}{g}\right). \quad (6.14)$$

The trace of the extrinsic curvature is used to compute the Gibbons-Hawking term I_{GH}. For I_2, we also need the intrinsic curvature on the boundary, $\mathcal{R} = \frac{2}{r^2}f(4-hf^3)$. Adding all contributions according to (4.9), we finally arrive at

$$\begin{aligned} I_{\text{complete}} &= \frac{\beta\pi}{4G_5}\Big[\frac{3}{8}\ell^2 - \frac{\alpha_2}{2\ell^2} + \frac{1}{\ell^2}\Big(\frac{\mu_1^2(\mu_2-\mu_3)^2+\mu_2^2\mu_3^2}{(\mu_1-\mu_2)(\mu_1-\mu_3)}\log\mu_1 \\ &\quad + \frac{\mu_2^2(\mu_1-\mu_3)^2+\mu_1^2\mu_3^2}{(\mu_2-\mu_3)(\mu_2-\mu_3)}\log\mu_2 + \frac{\mu_3^2(\mu_1-\mu_2)^2+\mu_1^2\mu_2^2}{(\mu_3-\mu_1)(\mu_3-\mu_2)}\log\mu_3\Big)\Big]. \end{aligned} \quad (6.15)$$

6.2 The boundary stress tensor

We now proceed to compute the boundary stress tensor and to extract the ADM energy and angular momentum of this solution. In fact, the result is rather simple, and we find

$$\begin{aligned} T_{00}^{\text{complete}} &= \frac{1}{8\pi G_5}\frac{\ell}{r^4}\left(\frac{3}{8}\ell^2+\alpha_1+\frac{3\alpha_2}{2\ell^2}+\frac{2\alpha_3}{\ell^4}\right), \\ T_{03}^{\text{complete}} &= \frac{1}{8\pi G_5}\frac{\epsilon}{\ell r^4}\left(\alpha_2+\frac{2\alpha_3}{\ell^2}\right). \end{aligned} \quad (6.16)$$

In addition, $T_{11} = T_{22}$ and T_{33} are non-vanishing, but do not contribute to conserved quantities.

Taking into account (2.6), the conserved ADM energy and angular momentum are

$$\begin{aligned} E &= \frac{\pi}{4G_5}\left(\frac{3}{8}\ell^2+\alpha_1+\frac{3\alpha_2}{2\ell^2}+\frac{2\alpha_3}{\ell^4}\right), \\ J &= \frac{\epsilon\pi}{8G_5\ell}\left(\alpha_2+\frac{2\alpha_3}{\ell^2}\right). \end{aligned} \quad (6.17)$$

These expressions agree with those obtained by Gutowski and Reall [6] using the methods of Ashtekar and Das [7], provided one relates the ADM energy E and the Ashtekar and Das mass M through

$$E = M + \frac{3\pi\ell^2}{32G_5}. \quad (6.18)$$

The latter contribution is identified as the Casimir energy, and verifies the prediction of Gutowski and Reall.

7 Discussion

Computing black hole energies using the boundary stress tensor method is natural in the AdS/CFT context. What we have shown here is that, by incorporating a ϕ^2 counterterm, we are able to derive the expected ADM energies for the non-rotating R-charged black holes, (5.9), and the rotating BPS solutions, (6.17). In the former case, this finite counterterm removes a non-linear charge contribution to the energy, while in the latter case, it modifies but does not remove the non-linearities.

For the case of the non-rotating black holes, the linear mass relation (5.9) verifies the result of [9]. As this was the basis of the thermodynamic exploration of R-charged black holes in [10], we have shown that the standard results follow naturally from the boundary stress tensor prescription, provided appropriate finite counterterms are incorporated. The mass of rotating Einstein-Maxwell AdS_5 black holes was also examined using the boundary stress tensor method in [11]. The result of [11] ought to be generalizable to the STU model after inclusion of the appropriate ϕ^2 counterterm.

Of course, as we have indicated, the energy computed in this manner is not unique, and depends on the nature of finite counterterms used in regulating the boundary stress tensor. This fact is understood in terms of having to specify a particular counterterm prescription with which to work with; in a field theory language, this is simply the scheme dependence of standard renormalization. Although the energy, as so defined, is ambiguous up to finite counterterms, physical quantities in the dual field theory must always be well defined. However, in practice, what is and is not scheme dependent is often a subtle issue, and separating the two may require care.

In order to deal with this ambiguity, it is natural to impose some additional symmetry requirements on the regularization procedure. In the present case, our desire to expose a linear BPS-like relation between mass and R-charge in the dual CFT has led us to postulate the addition of the finite ϕ^2 counterterm in (4.9). In fact, such a counterterm can be motivated by Hamilton-Jacobi theory, and can be seen as a necessity for the preservation of supersymmetry in the boundary theory. Note, also, that for the case of AdS_4 with scalars, the ϕ^2 counterterm is no longer optional, but necessary to render the action finite. This connection to the Hamilton-Jacobi approach for matter coupled gravity systems will be explored in a subsequent publication [12].

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